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Spinor Structure of P -Oriented Space, Kustaanheimo-Stiefel and Hopf Bundle – Connection between Formalisms

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In the work some relations between three techniques, Hopf's bundle, Kustaanheimo-Stiefel's bundle, 3-space with spinor structure have been examined. The spinor space is viewed as a real space that is minimally (twice as much) extended in comparison with an ordinary vector 3-space: at this instead of 2π -rotation now only 4π -rotation is taken to be the identity transformation in the geometrical space. With respect to a given P -orientation of an initial unextended manifold, vector or pseudovector one, there may be constructed two different spatial spinors, ξ and η , respectively. By definition, those spinors provide us with points of the extended space models, each spinor is in the correspondence $2 \rightarrow 1$ with points of a vector space. For both models an explicit parametrization of the spinors ξ and η by spherical and parabolic coordinates is given, the parabolic system turns out to be the most convenient for simple defining spacial spinors. Fours of real-valued coordinates by Kustaanheimo-Stiefel, U_a and V_a , real and imaginary parts of complex spinors ξ and η respectively, obey two quadratic constraints. So that in both cases, there exists a Hopf's mapping from the part of 3-sphere S_3 into the entire 2-sphere S_2 . Relation between two spacial spinor is found: $\eta = (\xi - i \sigma^2 \xi^*)/\sqrt{2}$, in terms of Kustaanheimo-Stiefel variables U_a and V_a it is a linear transformation from $SO(4.R)$, which does not enter its sub-group generated by $SU(2)$ -rotation over spinors.

1 Introduction

In the literature [1-66] there exist three terminologically different, but close in mathematical technique, approaches. These are spinor space structure¹, Hopf's bundle [5], Kustaanheimo-Stiefel bundle [9,10].

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¹See for example [29,61-66], however an idea itself to employ in all parts exceptionally spinors in place of vectors may be seen in the very creating the quantum theory; see, in particular, Cartan's and Weyl's works [2-4].

It is interesting to note that before creating the Cartan's group theory of spinors Darboux' lectures [1] contain description of a map which in present-day terminology can be referred to a spinor representation for spinors, or in other words to Hopf's and Kustaanheimo-Stifel formalism.

Differences between three mentioned approaches consist mainly in conceptual accents.

In the Hopf's technique it is suggested to use in all parts only complex spinors ξ and conjugate ξ^* instead of real-valued vector (tensor) quantities. This approach is used mainly for a vector that can be associated with a pseudo vector space model (x, y, z) ; at this a phase factor at ξ does not influence real-valued vector coordinates (x, y, z) .

In the Kustaanheimo-Stifel approach we are to use four real-valued coordinates, from which by means of definite bi-linear functions Cartesian coordinates (x, y, z) can be formed up. These Kustaanheimo-Stifel's four variables are real and imaginary parts of two spinor components. The known spinor invariant under $SU(2)$, being transformed to these variables, becomes the sum of four squared quantities, so that we can associate spinor technique with geometry of 3-dimensional Riemann space S_3 .

In essence, the Kustaanheimo-Stifel's approach is elaboration of complex spinor, ξ and ξ^* , Hopf's technique in term of four real-valued variables. At this we are able to hide (in appearance only) the presence in the formalism of the non-analytical operation of complex conjugation.

Spinor space structure, formalism developed in present work, also exploits possibilities given by spinors to construct 3-vectors, however the emphasis is taken to doubling the set of spatial points so that we get an extended space model that is called a space with spinor structure [61-66]. In such an extended space, in place of 2π -rotation, only 4π -rotation transfers the space into itself. In the present work this extended space model is investigated with the help of curvilinear coordinates [66]. At this we consistently distinct two sorts of spatial spinors: which are referred to a vector and pseudo vector models respectively. Else one point; as shown, the procedure itself of doubling the manifold can be realized easier when for parameterizing the space some curvilinear coordinate system is used in contrast with the Cartesian coordinates. In the work we consider in detail two coordinates systems: spherical and parabolic ones.

It should be noted that because all three technique exploit much the same mathematical formalism, always it is possible transportation results from one technique to another.

The main purpose of the present work is to describe some relations between these three approaches. In particular, it is shown the following: the structure of spinor space model can be considered as minimization of (Hopf and Kustaanheimo-Stifel) mapping in the sense that we can eliminate from 3-dimension set $U_1^2 + U_2^2 + U_3^2 + U_4^2 = 2r$, Riemann sphere S_3 , all the redundant points so that the remaining 2-dimension set $\bar{S}_3 \in S_3$ is mappable into 2-sphere $x^2 + y^2 + z^2 = r^2$ by taking the rule $2 \rightarrow 1$. In addition, different spinor space model are established in dependence of P -orientation of an initial vector space.

2 Spinor extension of a pseudo vector space model

Extending the vector space with the aim to describe its spinor structure may be done on the base of the known decomposition of two rank spinor (we mean spinor under Cartan's extended

unitary group $SU(2)$ [1-3]

$$\begin{aligned}\xi \otimes \xi^* &= (r + x_j \sigma^j) , \\ x_j &= \frac{1}{2} \text{sp} [\sigma^j (\xi \otimes \xi^*)] = \frac{1}{2} \xi^+ \sigma^j \xi , \\ r &= \frac{1}{2} \text{sp} (\xi \otimes \xi^*) = \frac{1}{2} \xi^+ \xi = \text{inv} .\end{aligned}\tag{1}$$

Under continuous transformations from $SU(2)$ group the coefficients x_j in (1) transform as 3-vector representation [60]

$$\begin{aligned}B(n) &= I n_4 - i \sigma^j n_j , \\ n_4^2 + n_j n_j &= 1 , \quad \xi' = B(n) \xi , \\ O(n) &= I + 2 [n_4 \vec{n}^\times + (\vec{n}^\times)^2] , \\ x'_k &= O_{kl}(n) x_l , \quad (\vec{n}^\times)_{kl} = -\epsilon_{klj} n_j .\end{aligned}$$

Indeed,

$$x'_k = \frac{1}{2} \xi'^+ \sigma^k \xi' = \frac{1}{2} \xi^+ [B^+(n) \sigma^k B(n)] \xi = \frac{1}{2} \xi^+ O_{kl}(n) \sigma^l \xi = O_{kl}(n) x_l .\tag{2}$$

Under Cartan's spinor P -reflection, at any intrinsic parity $\delta = \pm 1$ of spinor ξ , the quantity x_j transforms as a pseudo vector:

$$\xi' = \delta J \xi , \quad J = \begin{vmatrix} i & 0 \\ 0 & i \end{vmatrix} , \quad x'_j = +x_j .\tag{3}$$

Solving eq. (1) with respect to spinor ξ that parameterizes an initial pseudo vector space in accordance with the rule $2 \rightarrow 1$, lead us to the following formulas (take notice to $\phi \in [-2\pi, +2\pi]$)

$$\begin{aligned}\xi(x_1, x_2, x_3) &= \begin{vmatrix} U_1 + i U_2 \\ U_3 + i U_4 \end{vmatrix} = \begin{vmatrix} \sqrt{r+x_3} & e^{-i\phi/2} \\ \sqrt{r-x_3} & e^{+i\phi/2} \end{vmatrix} , \\ r &= \sqrt{x_1^2 + x_2^2 + x_3^2} , \quad e^{i\phi} = \frac{x_1 + i x_2}{\sqrt{x_1^2 + x_2^2}} ,\end{aligned}\tag{4}$$

real-valued U_1, \dots, U_4 coincide with Kustaanheimo-Stifel variables.

As we employ spherical coordinates, the space spinor ξ will take the form [66]

$$\xi(r, \theta, \phi) = \begin{vmatrix} \sqrt{r(1+\cos\theta)} & e^{-i\phi/2} \\ \sqrt{r(1-\cos\theta)} & e^{i\phi/2} \end{vmatrix} , \quad \phi \in [-2\pi, +2\pi] .\tag{5}$$

Thus, four Kustaanheimo-Stifel coordinates U_a are functions of three spherical ones (r, θ, ϕ) , so that we are to expect some additional relationship between these four parameters.

Further we get to

$$\frac{1}{2} (\xi^+ \xi) = \frac{1}{2} (U_1^2 + U_2^2 + U_3^2 + U_4^2) = r ,\tag{6}$$

$$\frac{1}{2} (\xi^+ \sigma^1 \xi) = U_1 U_3 + U_2 U_4 = r \sin \theta \cos \phi = x_1 ,$$

$$\frac{1}{2} (\xi^+ \sigma^2 \xi) = U_1 U_4 - U_2 U_3 = r \sin \theta \sin \phi = x_2 ,$$

$$\frac{1}{2} (\xi^+ \sigma^3 \xi) = \frac{1}{2} (U_1^2 + U_2^2 - U_3^2 - U_4^2) = r \cos \theta = x_3 .\tag{7}$$

In other words, Cartesian coordinates x_1, x_2, x_3 have been expressed as bi-linear combinations of variables U_1, \dots, U_4 . It should be noted that the real-valued parameters U_a taken from (4) and (5) obey the identity

$$U_1 U_4 + U_2 U_3 = 0 . \quad (8)$$

which points out that we have dealings with the Hopf mapping $S_3 \implies S_2$ with the rule $2 \longrightarrow 1$ from special sub-set in 3-sphere into the full 2-sphere:

$$\{ U_a U_a = 2r, U_1 U_4 + U_2 U_3 = 0 \} \implies \{ x_1^2 + x_2^2 + x_3^2 = r^2 \} . \quad (9)$$

3 Spinor extension of a vector space model

Constructing a spinor model for a vector space can be done in the same line but on the base of the formula (decomposition of direct product of a spinor by itself in terms of Pauli matrices)

$$\eta \otimes \eta = (a_j + i x_j) \sigma^j \sigma^2, \quad (a_j + i x_j) = \frac{1}{2} \text{sp} [\sigma^2 \sigma^j (\eta \otimes \eta)] . \quad (10)$$

It is easier verified that $(a_j + i x_j)$ transforms as a 3-vector under $SU(2)$ group (its real and imaginary parts transform independently by means of real-valued rotation matrix). Indeed

$$a'_j + i x'_j = \frac{1}{2} \tilde{\eta}' \sigma^2 \sigma^j \eta' = \frac{1}{2} \tilde{\eta} [\tilde{B}(n) \sigma^2 \sigma^j B(n)] \eta ,$$

from here, with the known identity for spinor transformations [60]

$$\tilde{B}(n) \sigma^2 = \sigma^2 B^{-1}(n) , \quad B^{-1}(n) \sigma^j B(n) = O_{jl}(n) \sigma^l ,$$

we get

$$(a'_j + i x'_j) = O_{jl}(n) (a_l + i x_l) .$$

As seen, under Cartan's spinor P – reflection (at any intrinsic parity $\delta = \pm 1$ of the spinor η), the quantity $(a_j + i x_j)$ behaves as a vector:

$$\eta' = \delta J \eta, \quad J = \begin{vmatrix} i & 0 \\ 0 & i \end{vmatrix}, \quad (a'_j + i x'_j) = - (a_j + i x_j) . \quad (11)$$

Solving eq. (10) with respect to η , which parameterizes the vector space x_j by scheme $2 \longrightarrow 1$, we get to [66]

$$\eta^\sigma(x_1, x_2, x_3) = \begin{vmatrix} V_1 + i V_2 \\ V_3 + i V_4 \end{vmatrix} = \begin{vmatrix} \sqrt{x - \rho} & \sigma e^{-i\phi/2} \\ \sqrt{x + \rho} & e^{+i\phi/2} \end{vmatrix} ,$$

$$\rho = \sqrt{x_1^2 + x_2^2}, \quad e^{i\phi} = \frac{x_1 + i x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \phi \in [-2\pi, +2\pi] ; \quad (12)$$

where $\sigma = +1$ corresponds to upper half-space $x_3 > 0$, and $\sigma = -1$ corresponds to the half-space $x_3 < 0$.

In particular, spinor η looks in spherical coordinates as follows [66]

$$\eta^\sigma(r, \theta, \phi) = \begin{vmatrix} \sqrt{r(1 - \sin \theta)} & \sigma e^{-i\phi/2} \\ \sqrt{r(1 + \sin \theta)} & e^{i\phi/2} \end{vmatrix}. \quad (13)$$

This representation of the spinor can be easily verified by direct calculation. Indeed

$$\begin{aligned} \frac{1}{2} \text{sp} [\sigma^1(\eta^\sigma \otimes \eta^\sigma)\sigma^2] &= a_1 + i x_1 = -r \sin \phi + i r \sin \theta \cos \phi, \\ \frac{1}{2} \text{sp} [\sigma^2(\eta^\sigma \otimes \eta^\sigma)\sigma^2] &= a_2 + i x_2 = r \cos \phi + i r \sin \theta \sin \phi, \\ \frac{1}{2} \text{sp} [\sigma^3(\eta^\sigma \otimes \eta^\sigma)\sigma^2] &= a_3 + i x_3 = 0 + i r \cos \theta. \end{aligned} \quad (14)$$

The identity $a_3 = 0$ is not accidental, it takes place at any coordinate system. Indeed, with the help of eq. (12) we get

$$(\eta \otimes \eta) = \begin{vmatrix} (x - \rho)e^{-i\gamma} & \sqrt{x^2 - \rho^2} \sigma \\ \sqrt{x^2 - \rho^2} \sigma & (x + \rho)e^{+i\gamma} \end{vmatrix} = \begin{vmatrix} (x - \rho)e^{-i\gamma} & x_3 \\ x_3 & (x + \rho)e^{+i\gamma} \end{vmatrix}$$

and further

$$\begin{aligned} a_1 + i x_1 &= -x \sin \gamma + i \rho \cos \gamma, \\ a_2 + i x_2 &= +x \cos \gamma + i \rho \sin \gamma, \\ a_3 + i x_3 &= 0 + i \sqrt{x^2 - \rho^2} \sigma. \end{aligned} \quad (15)$$

Let us produce the formulas connecting Cartesian coordinates x_j from (14) with corresponding Kustaanheimo-Stiefel variables. The problem is reduced to expressing $a_j + i x_j$ in terms of real-valued parameters V_1, \dots, V_4 :

$$\begin{aligned} & a_1 + i x_1 \quad (16) \\ &= V_1 V_2 - V_3 V_4 + i \frac{-V_1^2 + V_2^2 + V_3^2 - V_4^2}{2}, \\ & a_2 + i x_2 \\ &= \frac{V_1^2 - V_2^2 + V_3^2 - V_4^2}{2} + i(V_1 V_2 + V_3 V_4), \\ & i x_3 = -V_1 V_4 - V_2 V_3 + i(V_1 V_3 - V_2 V_4). \end{aligned}$$

Additionally we can easily produce the formula for r^2 in terms V_1, \dots, V_4 :

$$x_1^2 + x_2^2 + x_3^2 = \frac{1}{4} (V_1^2 + V_2^2 + V_3^2 + V_4^2)^2. \quad (17)$$

This means that again we have dealings with the Hopf map from a party of the sphere S_3 into the full 2-sphere S_2 :

$$\{ V_a V_a = 2r, \ V_1 V_4 + V_2 V_3 = 0 \} \implies \{ x_1^2 + x_2^2 + x_3^2 = r^2 \}. \quad (18)$$

4 Parabolic coordinates and determining spatial spinors

It is turned out that the known parabolic coordinates plays a special role in defining spatial spinor ξ and η . First let us consider the pseudo vector model (see also in [66]). Let the spatial spinor ξ be given by its natural parameters (N, M, ϕ) :

$$\xi = \begin{vmatrix} N e^{-i\phi/2} \\ M e^{+i\phi/2} \end{vmatrix}, \quad \phi \in [-2\pi, +2\pi], \quad M \in [0, +\infty), \quad N \in [0, +\infty), \quad . \quad (19)$$

It is easily to show that (N, M, ϕ) coincides with the known parabolic coordinates (to the case of vector model there corresponds the domain $\gamma \in [0, +2\pi]$). To this end, with the help of

$$\xi \otimes \xi^* = \begin{vmatrix} N^2 & NM e^{-i\phi} \\ NM e^{+i\phi} & M^2 \end{vmatrix}$$

one finds

$$\begin{aligned} x_1 &= \frac{1}{2} \text{sp} [\sigma^1 \xi \otimes \xi^*] = NM \cos \phi, \\ x_2 &= \frac{1}{2} \text{sp} [\sigma^2 \xi \otimes \xi^*] = NM \sin \phi, \\ x_3 &= \frac{1}{2} \text{sp} [\sigma^3 \xi \otimes \xi^*] = \frac{N^2 - M^2}{2}, \\ r &= \frac{1}{2} \text{sp} [\xi \otimes \xi^*] = \frac{N^2 + M^2}{2}, \end{aligned} \quad (20)$$

From this, taking into account determining relations for parabolic coordinates

$$x_1 = \xi \eta \cos \phi, \quad x_2 = \xi \eta \sin \phi, \quad x_3 = \frac{\xi^2 - \eta^2}{2}, \quad r = \frac{\xi^2 + \eta^2}{2}, \quad (21)$$

we may conclude that spinor ξ natural parameters (N, M, ϕ) can be identified with (ξ, η, ϕ) . The variable U_a are given as by

$$U_1 = N \cos \frac{\phi}{2}, \quad U_2 = -N \sin \frac{\phi}{2}, \quad U_3 = M \cos \frac{\phi}{2}, \quad U_4 = M \sin \frac{\phi}{2}. \quad (22)$$

From this it follows

$$U_1 U_4 + U_2 U_3 = 0. \quad (23)$$

Now let us proceed to the second model. Let us designate parabolic coordinates as (N, M, ϕ) . With the help of

$$\begin{aligned} x_1 &= NM \cos \phi, & x_2 &= NM \sin \phi, \\ x_3 &= \frac{N^2 - M^2}{2}, & r &= \frac{N^2 + M^2}{2}, & \rho &= N M, \end{aligned}$$

we get

$$(\sigma) (+\sqrt{x-\rho}) = \frac{N-M}{\sqrt{2}}, \quad +\sqrt{x+\rho} = \frac{N+M}{\sqrt{2}}. \quad (24)$$

Therefore, spatial spinor η looks as (take notice that $\sigma = \pm 1$ disappears)

$$\eta = \frac{1}{\sqrt{2}} \begin{vmatrix} (N - M) e^{-i\phi/2} \\ (N + M) e^{+i\phi/2} \end{vmatrix}. \quad (25)$$

From this it follows

$$\begin{aligned} a_1 + ix_1 &= -\frac{N^2 + M^2}{2} \sin \phi + iNM \cos \phi, \\ a_2 + ix_2 &= +\frac{N^2 + M^2}{2} \cos \phi + iNM \sin \phi, \\ a_3 + ix_3 &= 0 + i\frac{N^2 - M^2}{2}. \end{aligned} \quad (26)$$

The variables V_1, \dots, V_4 are given by

$$\begin{aligned} V_1 &= \frac{N - M}{\sqrt{2}} \cos \frac{\phi}{2}, \quad V_2 = -\frac{N - M}{\sqrt{2}} \sin \frac{\phi}{2}, \\ V_3 &= \frac{N + M}{\sqrt{2}} \cos \frac{\phi}{2}, \quad V_4 = +\frac{N + M}{\sqrt{2}} \sin \frac{\phi}{2}. \end{aligned} \quad (27)$$

5 Connection between the variables U_a and V_a

Comparing (27) with (22), we readily establish relationships:

$$\begin{aligned} V_1 &= \frac{U_1 - U_3}{\sqrt{2}}, \quad V_2 = \frac{U_2 + U_4}{\sqrt{2}}, \\ V_3 &= \frac{U_1 + U_3}{\sqrt{2}}, \quad V_4 = \frac{-U_2 + U_4}{\sqrt{2}}, \end{aligned} \quad (28)$$

or in a matrix form

$$\begin{vmatrix} V_4 \\ V_1 \\ V_2 \\ V_3 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} U_4 \\ U_1 \\ U_2 \\ U_3 \end{vmatrix}. \quad (29)$$

Returning with (29) to (12), for $\eta(V_a)$ one produces (U_1):

$$\eta = \begin{vmatrix} V_1 + i V_2 \\ V_3 + i V_4 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} (U_1 - U_3) + i (+U_2 + U_4) \\ (U_1 + U_3) + i (-U_2 + U_4) \end{vmatrix}, \quad (30)$$

with the relations

$$\begin{aligned} U_a U_a &= 2r, & V_a V_a &= 2r, \\ U_1 U_4 + U_2 U_3 &= 0, & V_1 V_4 + V_2 V_3 &= 0. \end{aligned} \quad (31)$$

Therefore, an explicit form of two spatial spinors ξ and η differs when we use ordinary curvilinear coordinates (cartesian, spherical, parabolic) and also when we use Kustaanheimo-Stifel variables. This means that just these explicit forms of two sorts of spinors describes essential difference of two spinor space models.

Relation between ξ and η can be readily expressed in terms of complex spinor themselves. Indeed, with the help of

$$\eta = \frac{1}{\sqrt{2}} \begin{vmatrix} (N - M) e^{-i\phi/2} \\ (N + M) e^{+i\phi/2} \end{vmatrix}, \xi = \begin{vmatrix} N e^{-i\phi/2} \\ M e^{+i\phi/2} \end{vmatrix}.$$

one produces

$$\eta = \frac{1}{\sqrt{2}} (\xi - i \sigma^2 \xi^*) . \quad (32)$$

inverse relation looks as

$$\xi = \frac{1}{\sqrt{2}} (\eta - i \sigma^2 \eta^*) . \quad (33)$$

The most significant point in connection with (32) and (33) is that transition from one type of spinor to another includes the complex conjugation. This points out that there does not exist analytical map (in the sense of complex variable function theory) which could be able to relate these two spinor ξ and η . This peculiarity in relation ξ to η is much disguised when we employ the real-valued variables U_a and V_a .

Let us consider some additional properties of the relation $V_a = S_{ab} U_b$:

$$\begin{vmatrix} V_4 \\ V_1 \\ V_2 \\ V_3 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} U_4 \\ U_1 \\ U_2 \\ U_3 \end{vmatrix}. \quad (34)$$

It is readily checked that

$$\det S = +1 : \implies S \in SO(4.R) . \quad (35)$$

Therefore, the matrix S belongs to the group $SO(4.R)$. There exist six elementary rotations in this group, in the planes $2-3, 3-1, 1-2$ and $4-1, 3-2, 4-3$:

$$S_{2-3}(\phi_1) = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi_1 & -\sin \phi_1 \\ 0 & 0 & \sin \phi_1 & \cos \phi_1 \end{vmatrix}, \quad S_{3-1}(\phi_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi_2 & 0 & -\sin \phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin \phi_2 & 0 & \cos \phi_2 \end{vmatrix},$$

$$S_{1-2}(\phi_3) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi_3 & -\sin \phi_3 & 0 \\ 0 & \sin \phi_3 & \cos \phi_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad S_{4-1}(\beta_1) = \begin{vmatrix} \cos \beta_1 & -\sin \beta_1 & 0 & 0 \\ \sin \beta_1 & \cos \beta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$S_{4-2}(\beta_2) = \begin{vmatrix} \cos \beta_2 & 0 & -\sin \beta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sin \beta_2 & 0 & \cos \beta_2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad S_{4-3}(\beta_3) = \begin{vmatrix} \cos \beta_3 & 0 & 0 & -\sin \beta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin \beta_3 & 0 & 0 & \cos \beta_3 \end{vmatrix}. \quad (36)$$

Let us clarify the structure of the matrix S in (34) in terms of elementary rotations (36):

$$S = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}. \quad (37)$$

that is

$$S = S_{4-2} \left(\frac{\pi}{2} \right) S_{3-1} \left(\frac{\pi}{4} \right) = S_{4-2} \left(\frac{\pi}{4} \right) S_{3-1} \left(\frac{\pi}{2} \right). \quad (38)$$

Else one additional test is possible. Let us show that the matrix (34), though being an element of the group $SO(4.R)$, nevertheless it does not belong to its sub-group generated by transformations from $SU(2)$, acting on spinor ξ .

Generally speaking, in the light of eq. (5.3) the reply to this question is evident in advance. However it is important in other context. In [55-58], instead of the Kustaanheimo-Stifel variables U_a it was constructed a complete set of such variables (W_a, \vec{A}) ; more details see in Appendix A.

Transition to the set (W_a, \vec{A}) is based on a simple geometrical idea. It can be clarified on the familiar case of spherical coordinates. Let certain spherical coordinates be determined by

$$x = r \sin \theta \sin \phi, y = r \sin \theta \cos \phi, z = r \cos \theta$$

however it is understandable that you may determine other spherical coordinates (r', θ', ϕ') related to respective cartesian coordinates (x', y', z') :

$$x' = r' \sin \theta' \sin \phi', y' = r' \sin \theta' \cos \phi', z' = r' \cos \theta'.$$

As long as two cartesian sets may be referred to each other through certain rotation

$$x'_j = O_{ji}(c) x_i, \quad c = (c_4, c_j),$$

spherical sets (r, θ, ϕ) and (r', θ', ϕ') may be related as well. Explicit formulas establishing that connection are going to be not very simple and interesting. However, the same idea, being applied to introducing new curvilinear coordinates, Kustaanheimo-Stifel variables (U_a) , turns out to be useful because the formulas relating different sets (U_a) and (U'_a) are quite simple ones.

As shown (see [58] and also Appedix A, B of the present work), all those new variables (U'_a) are generated from initial ones (U_a) through the matrices of $SU(2)$.

Let us consider action of $SU(2)$ elements on spatial spinor ξ in terms of real-valued variables. To this end, equation

$$\xi' = B(c) \xi \quad (39)$$

should be translated to real form. With the help of

$$B(c) = \begin{vmatrix} c_4 - ic_3 & -c_2 - ic_1 \\ c_2 - ic_1 & c_4 + ic_3 \end{vmatrix}, \quad \xi = \begin{vmatrix} U_1 + i U_2 \\ U_3 + i U_4 \end{vmatrix}, \quad \xi' = \begin{vmatrix} U'_1 + i U'_2 \\ U'_3 + i U'_4 \end{vmatrix} \quad (40)$$

we get to

$$\begin{vmatrix} U'_4 \\ U'_1 \\ U'_2 \\ U'_3 \end{vmatrix} = \begin{vmatrix} c_4 & -c_1 & c_2 & c_3 \\ c_1 & c_4 & c_3 & -c_2 \\ -c_2 & -c_3 & c_4 & -c_1 \\ -c_3 & c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} U_4 \\ U_1 \\ U_2 \\ U_3 \end{vmatrix}. \quad (41)$$

Let us write down the relation between $\xi(U_a)$ $\eta(V_a)$, in the real form too:

$$\begin{vmatrix} V_4 \\ V_1 \\ V_2 \\ V_3 \end{vmatrix} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \begin{vmatrix} U_4 \\ U_1 \\ U_2 \\ U_3 \end{vmatrix}. \quad (42)$$

It is seen that the transformation (42) cannot be obtained from set of matrices (41) at any choice of (c_4, c_j) . Therefore, the variables U_a and V_a are essentially different.

6 Conclusion

In the work some relations between three techniques, Hopf's bundle, Kustaanheimo-Stiefel's bundle, 3-space with spinor structure have been examined. The spinor space is viewed as a real space that is minimally (twice as much) extended in comparison with an ordinary vector 3-space: at this instead of 2π -rotation now only 4π -rotation is taken to be the identity transformation in the geometrical space. With respect to a given P -orientation of an initial unextended manifold, vector or pseudovector one, there may be constructed two different spatial spinors, ξ and η , respectively. By definition, those spinors provide us with points of the extended space models, each spinor is in the correspondence $2 \rightarrow 1$ with points of a vector space. For both models an explicit parametrization of the spinors ξ and η by spherical and parabolic coordinates is given, the parabolic system turns out to be the most convenient for simple defining spacial spinors. Fours of real-valued coordinates by Kustaanheimo-Stiefel, U_a and V_a , real and imaginary parts of complex spinors ξ and η respectively, obey two quadratic constraints. So that in both cases, there exists a Hopf's mapping from the part of 3-sphere S_3 into the entire 2-sphere S_2 . Relation between two spacial spinor is found: $\eta = (\xi - i \sigma^2 \xi^*)/\sqrt{2}$, which in terms of Kustaanheimo-Stiefel variables U_a and V_a is a linear transformation from $SO(4.R)$, which does not enter its sub-group generated by $SU(2)$ -rotation over spinors.

In addition to the main line of the present work, Appendix B gives some preliminary analysis to the problem of finding $SU(2)$ rotation which connects two different 2-spinors. It is shown that the use of spinor description eliminates from the formalism the concept of a small group.

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Appendix A. Kustaanheimo-Stifel bundle and covariance

In [58] it was performed generalization² for a standard Kustaanheimo-Stifel formalism:

$$\begin{aligned}\xi \otimes \xi^* &= (r + x_j \sigma^j) , \\ x_j &= \frac{1}{2} \text{sp} [\sigma^j (\xi \otimes \xi^*)] = \frac{1}{2} \xi^+ \sigma^j \xi , \\ r &= \frac{1}{2} \text{sp} (\xi \otimes \xi^*) = \frac{1}{2} \xi^+ \xi ;\end{aligned}\tag{A.1}$$

$$\xi = \begin{vmatrix} U_1 + i U_2 \\ U_2 + i U_4 \end{vmatrix} ;\tag{A.2}$$

$$\begin{aligned}x_1 &= U_1 U_3 + U_2 U_4 , \\ x_2 &= U_1 U_4 - U_2 U_3 , \\ 2 x_3 &= U_1^2 + U_2^2 - U_3^2 - U_4^2 , \\ 2 r &= U_1^2 + U_2^2 + U_3^2 + U_4^2 .\end{aligned}\tag{A.3}$$

To a set of U_a there corresponds a point in the 3-sphere of the radius $\sqrt{2r}$. Significant feature of the map (A.1)–(A.3) consists in the following: all the spinors ξ different in a phase factor $e^{i\alpha}$, $\alpha \in [0, 2\pi]$, will generate the same vector \vec{x} :

$$\xi \implies \vec{x} , \quad \alpha \in [0, 2\pi] , \quad e^{i\alpha} \xi \implies \vec{x} .\tag{A.4}$$

In terms of U_a this peculiarity looks as

$$\begin{aligned}U_a &\implies x_j , \\ \begin{vmatrix} \tilde{U}_1 \\ \tilde{U}_2 \end{vmatrix} &= \begin{vmatrix} \cos \alpha & -\sin \alpha \\ +\sin \alpha & \cos \alpha \end{vmatrix} \begin{vmatrix} U_1 \\ U_2 \end{vmatrix} , \\ \begin{vmatrix} \tilde{U}_3 \\ \tilde{U}_4 \end{vmatrix} &= \begin{vmatrix} \cos \alpha & -\sin \alpha \\ +\sin \alpha & \cos \alpha \end{vmatrix} \begin{vmatrix} U_3 \\ U_4 \end{vmatrix} , \\ \tilde{U}_a &\implies x_j .\end{aligned}\tag{A.5}$$

It should be noted, however, that those $U(1)$ -transformations will take away the points U_a from the surface $U_1 U_4 + U_2 U_3 = 0$. Indeed,

$$\tilde{U}_1 \tilde{U}_4 + \tilde{U}_2 \tilde{U}_3 = \sin 2\alpha (U_1 U_3 - U_2 U_4) \neq 0 .\tag{A.6}$$

²Below this approach will be given with some differences in notation.

Only four particular values of $e^{i\alpha}$ leave the surface $U_1U_4 + U_2U_3 = 0$ the same:

$$e^{i\alpha} = +1, -1, +i, -i . \quad (\text{A.7})$$

It turned out to be helpful [35,58] if one will employ a factorized form for U_a :

$$\begin{aligned} U_a &= \sqrt{2r} \, u_a, \quad u_1^2 + u_2^2 + u_3^2 + u_4^2 = +1 , \\ u_a &= \frac{U_a}{\sqrt{2r}} = \frac{U_a}{\sqrt{U_1^2 + U_2^2 + U_3^2 + U_4^2}} . \end{aligned} \quad (\text{A.8})$$

At this the spinor ξ from (A.2) becomes

$$\xi = \sqrt{2r} \begin{vmatrix} u_1 + i u_2 \\ u_3 + i u_4 \end{vmatrix} , \quad (\text{A.9})$$

and relations (A.1) and (A.3) give

$$\vec{x} = \frac{1}{2} \xi^+ \vec{\sigma} \xi : \implies \vec{n} = \frac{1}{\sqrt{2r}} \xi^+ \vec{\sigma} \frac{1}{\sqrt{2r}} \xi ,$$

and

$$\begin{aligned} n_1 &= \frac{x_1}{r} = 2 (u_1 u_3 + u_2 u_4) , \\ n_2 &= \frac{x_2}{r} = 2 (u_1 u_4 - u_2 u_3) , \\ n_3 &= \frac{x_3}{r} = u_1^2 + u_2^2 - u_3^2 - u_4^2 , \\ &u_1^2 + u_2^2 + u_3^2 + u_4^2 = +1 . \end{aligned} \quad (\text{A.10})$$

Accounting for the condition $u_a u_a = +1$, with the help of u_a one can construct a matrix from $SU(2)$ group:

$$B(u) = (u_4 I - i \sigma^j u_j) \in SU(2) .$$

On direct calculation one readily establishes [35,58] the formulas

$$\begin{aligned} B(u) \sigma^3 B^{-1}(u) &= \begin{vmatrix} -n_3 & n_1 + i n_2 \\ n_1 - i n_2 & +n_3 \end{vmatrix} \\ &= +n_1 \sigma^1 - n_2 \sigma^2 - n_3 \sigma^3 \equiv \sigma^1 (n_j \sigma^j) \sigma^1 . \end{aligned} \quad (\text{A.11})$$

It is better to rewrite the formula in the form (take notice that $\pm i \sigma^1 \in SU(2)$)

$$B(u) \sigma^3 B^{-1}(u) = (\pm i \sigma^1) (\vec{n} \vec{\sigma}) (\pm i \sigma^1)^{-1} . \quad (\text{A.12})$$

Two extremes terms in the right-hand part may be taken to the left-hand part and also they can be masked by special notation. Indeed,

$$(\pm i \sigma^1)^{-1} B(u) [(\pm i \sigma^1) \sigma^3 (\pm i \sigma^1)^{-1}] B^{-1}(u) (\pm i \sigma^1) = - \vec{n} \vec{\sigma} ;$$

from where, with ³

$$(\pm i\sigma^1)^{-1}B(u) (\pm i\sigma^1) = B(u_4; u_1, -u_2, -u_3) \equiv B(\hat{u}) , \quad (\text{A.13})$$

it follows

$$B(\hat{u})\sigma^3 B^{-1}(\hat{u}) = -\vec{n} \vec{\sigma} . \quad (\text{A.14})$$

This is only another form of the formulas (A.10) determining Kustaanheimo-Stiefel's normalized variables. It is easily established that the property (A.4) of the Kustaanheimo-Stiefel's mapping after translating to (A.14) is given by

$$\begin{aligned} B(\hat{u})e^{-i\alpha\sigma^3}\sigma^3 e^{i\alpha\sigma^3}B^{-1}(\hat{u}) &= -\vec{n} \vec{\sigma} \implies \\ B(\hat{\hat{u}})\sigma^3 B^{-1}(\hat{\hat{u}}) &= -\vec{n} \vec{\sigma} . \end{aligned} \quad (\text{A.15})$$

Let us find how in the form (A.14) will look transformations of variable u_a under $SU(2)$. Starting from

$$\begin{aligned} B(\hat{u})\sigma^3 B^{-1}(\hat{u}) &= -\vec{n} \vec{\sigma} : \implies \\ B(c)B(\hat{u})\sigma^3 B^{-1}(\hat{u})B^{-1}(c) &= -n_k B(c)\sigma^k B^{-1}(c) , \end{aligned}$$

and with the use of

$$B(c)B(\hat{u}) = B(\hat{u}') , \quad (\text{A.16})$$

we get

$$B(\hat{u}')\sigma^3 B^{-1}(\hat{u}') = -\sigma^l [O_{lk}(c) n_k] = -\vec{\sigma} \vec{n}' . \quad (\text{A.17})$$

Taking in mind additional α -rotations (A.15), the formula (A.16) should be modified by

$$B(c)B(\hat{u}) e^{-i\alpha\sigma^3} = B(\hat{u}') e^{-i\alpha'\sigma^3} . \quad (\text{A.18})$$

$SU(2)$ -transformation of the matrix (A.16) in the real-valued form will look

$$\begin{aligned} B(c)B(\hat{u}) = B(\hat{u}') &\implies \\ \begin{vmatrix} \hat{u}_4' \\ \hat{u}_1' \\ \hat{u}_2' \\ \hat{u}_3' \end{vmatrix} &= \begin{vmatrix} c_4 & -c_1 & c_2 & c_3 \\ c_1 & c_4 & c_3 & -c_2 \\ -c_2 & -c_3 & c_4 & -c_1 \\ -c_3 & c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} \hat{u}_4 \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{vmatrix} . \end{aligned} \quad (\text{A.19})$$

There exist useful possibility to modify the formula (A.19) as follows:

$$\begin{aligned} B(\hat{c})B(u) = B(u') &: \implies \\ \begin{vmatrix} u_4' \\ u_1' \\ u_2' \\ u_3' \end{vmatrix} &= \begin{vmatrix} c_4 & -c_1 & -c_2 & -c_3 \\ c_1 & c_4 & -c_3 & c_2 \\ c_2 & c_3 & c_4 & -c_1 \\ c_3 & -c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} . \end{aligned} \quad (\text{A.20})$$

³ That is \hat{u}_j is given by u_j after the 180° -rotation over the axis $(1, 0, 0)$.

The formula (A.14) readily displays remarkable relationship of Kustaanheimo-Stifel variables with matrices from orthogonal rotation group $SO(3.R)$. These 3-rotation in the frames of unitary group is given by

$$\begin{aligned} O(+c) &= O(-c) = I + 2 [c_4 \vec{c}^\times + (\vec{c}^\times)^2] , \\ (\vec{c}^\times)_{kl} &= -\epsilon_{klj} c_j . \end{aligned} \quad (\text{A.21})$$

Because of simple connection that to 3-vector parameter of $SO(3.R)$ [60]

$$\begin{aligned} \vec{C} &= \frac{\vec{c}}{c_4} , \quad O(+c) = O(-c) = I + 2 \frac{c_4 \vec{c}^\times + (\vec{c}^\times)^2}{c_4^2 + \vec{c}^2} \\ &= I + 2 \frac{\vec{C}^\times + (\vec{C}^\times)^2}{1 + \vec{C}^2} = O(\vec{C}) , \end{aligned} \quad (\text{A.22})$$

many advantages of this technique of the use of vector-parameter [60] preserves for unitary group as well. This relationships immediately appears as one takes eq. (A.14) in the context of the formula (see (2))

$$\begin{aligned} B^{-1}(c) \sigma^k B(c) &= O_{kj}(c) \sigma^j , \quad \implies \\ B(c) \sigma^k B^{-1}(c) &= \sigma^l O_{lk}(c) . \end{aligned} \quad (\text{A.23})$$

If $k = 3$ it takes on the form

$$B(c) \sigma^3 B^{-1}(c) = \sigma^j O_{j3}(c) . \quad (\text{A.24})$$

Comparing it with (A.14), we get to

$$\sigma^j O_{j3}(\hat{u}) = -\sigma^j n_j : \implies n_j = -O_{j3}(\hat{u}) . \quad (\text{A.25})$$

This formula may be checked by direct calculation with the use of explicit form of $O_{l3}(\hat{u})$. Starting from

$$O(c) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 2c_4 \begin{vmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{vmatrix} + 2 \begin{vmatrix} -c_2^2 - c_3^2 & c_1 c_2 & c_1 c_3 \\ c_2 c_1 & -c_3^2 - c_1^2 & c_2 c_3 \\ c_3 c_1 & c_3 c_2 & -c_2^2 - c_1^2 \end{vmatrix} ;$$

or

$$O(\hat{u}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 2u_4 \begin{vmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & -u_1 \\ u_2 & u_1 & 0 \end{vmatrix} + 2 \begin{vmatrix} -u_2^2 - u_3^2 & -u_1 u_2 & -u_1 u_3 \\ -u_2 u_1 & -u_3^2 - u_1^2 & u_2 u_3 \\ -u_3 u_1 & u_3 u_2 & -u_2^2 - u_1^2 \end{vmatrix} .$$

so that for third column we get

$$\begin{aligned} O_{13}(\hat{u}) &= -2(u_1 u_3 + u_2 u_4) , \\ O_{23}(\hat{u}) &= -2(u_1 u_4 - u_2 u_3) , \\ O_{33}(\hat{u}) &= -(u_1^2 + u_2^2 - u_3^2 - u_4^2) . \end{aligned} \quad (\text{A.26})$$

These relations coincide with (A.25). These relations are manifestly coordinate dependent; the task of giving them a covariant form was solved in [58]. As a base for this eq. (A.14) is taken. It is suggested instead of (A.14) to use a more general formula

$$B(\hat{w}) [\vec{A} \vec{\sigma}] B^{-1}(\hat{w}) = - \vec{n} \vec{\sigma} . \quad (\text{A.27})$$

Since in (A.27) dependence on A_j is linear, we may take the normalization condition $\vec{A}^2 = +1$. Also it is understandable that in place of (A.27) more general relation with 4-component $A^a = (A_4, \vec{A})$ might be taken, however this does not lead us to any interesting. Indeed

$$B(\hat{w}) [A_4 I - i \vec{A} \vec{\sigma}] B^{-1}(\hat{w}) = A_4 I - i(-\vec{\sigma} \vec{n}) .$$

The task is to establish relation [58] between any set (w_a, \vec{A}) in (A.27) and the initial set (u_a) . To this end, let us start with

$$B(\hat{w}) A^j \sigma^j B^{-1}(\hat{w}) = - n_j \sigma^j , \quad A_1^2 + A_2^2 + A_3^2 = +1 . \quad (\text{A.28})$$

Let $O(a), a = (a_4, a_j)$ be a matrix transforming the vector $\vec{A} = (A_1, A_2, A_3)$ into the vector $\vec{A}_{(+)} = (0, 0, +1)^4$:

$$A_{(+)}^j = O_{jl}(a) A^l , \quad A^j = O_{jl}^{-1}(a) A_{(+)}^l . \quad (\text{A.29})$$

With (A.29), eq. (A.28) becomes

$$B(\hat{w}) \sigma^j O_{jl}^{-1}(a) A_{(+)}^l B^{-1}(\hat{w}) = - n_j \sigma^j .$$

From this, with the use of

$$\sigma^j O_{jl}^{-1}(a) = B^{-1}(a) \sigma^l B(a) ,$$

one gets

$$B(\hat{w}) B^{-1}(a) \sigma^l A_{(+)}^l B(a) B^{-1}(\hat{w}) = - \vec{n} \vec{\sigma} ,$$

or

$$[B(\hat{w}) B(\bar{a})] \sigma^3 [B(\hat{w}) B(\bar{a})]^{-1} = - \vec{n} \vec{\sigma} . \quad (\text{A.30})$$

Comparing this with (A.14):

$$B(\hat{u}) \sigma^3 B^{-1}(\hat{u}) = - \vec{n} \vec{\sigma} ,$$

we arrive at condition connecting (u) and (w, \vec{A}) :

$$B(\hat{u}) = B(\hat{w}) B^{-1}(a), \quad B(\hat{w}) = B(\hat{u}) B(a) , \quad (\text{A.31})$$

⁴ There exists quite elaborated theory of such matrices in the frame of orthogonal group [59,60], in Section 8 its spinor counterpart will be considered.

or a more precise form

$$\begin{aligned} B(\hat{u})e^{-i\alpha\sigma^3} &= B(\hat{w}) B^{-1}(a)e^{-i\beta\sigma^3} \implies \\ B(\hat{w}) &= B(\hat{u})e^{-i(\alpha-\beta)\sigma^3} B(a) . \end{aligned} \quad (\text{A.32})$$

Mention that $a = (a_4, a_j)$ is defined by

$$\begin{aligned} A_{(+)}^j &= O_{jl}(a) A^l , \quad \vec{A} = (A_1, A_2, A_3) , \\ A_1^2 + A_2^2 + A_3^2 &= +1, \quad \vec{A}_0 = (0, 0, +1) . \end{aligned} \quad (\text{A.33})$$

Let us write down the relationship between $(u), (w)$ -variables and symmetry transformation over (u) :

$$B(\hat{w}) = B(\hat{u}) B(a) , \quad (\text{A.34})$$

$$B(\hat{u}') = B(c)B(\hat{u}) . \quad (\text{A.35})$$

One may identify $(\hat{w}) = (\hat{u}')$, so that a symmetry operation that generates (w) from (u) is found:

$$B(c) = B(\hat{u})B(a)B^{-1}(\hat{u}) , \quad (\text{A.36})$$

or its more precise form

$$B(c) = B(\hat{u})e^{-(\alpha-\beta)\sigma^3} B(a)[B(\hat{u})e^{-(\alpha-\beta)\sigma^3}]^{-1} .$$

For more understanding properties of (w, \vec{A}) -variables it is useful to perform some additional manipulation with formulas. The idea is to change the left-right side of (A.27) to the form of (A.14), at performing this we may expect appearance in the right-hand side a term with a new vector \vec{n}' that is specially rotated with respect to the initial \vec{n} .

To this end, let us express from (A.27) the combination $\vec{A} \vec{\sigma}$:

$$A^j \sigma^j = -n_j B^{-1}(\hat{w}) \sigma^j B(\hat{w}) . \quad (\text{A.37})$$

From where, with the use of

$$B^{-1}(\hat{w}) \sigma^j B(\hat{w}) = O_{jl}(\hat{w}) \sigma^l ,$$

we get

$$A^j \sigma^j = -n_j O_{jl}(\hat{w}) \sigma^l . \quad (\text{A.38})$$

Now let us carry out a special transformation giving on the left the term σ^3 . For this, let us multiply eq.(A.38) by $B(a)$ from the left and by $B^{-1}(a)$ from the right:

$$A^j B(a) \sigma^j B^{-1}(a) = -n_j O_{jl}(\hat{w}) B(a) \sigma^l B^{-1}(a) ,$$

and further

$$\sigma^k [O(a)_{kj} A^j] = -n_j O_{jl}(\hat{w}) O_{lk}^{-1}(a) \sigma^k .$$

Since

$$[O(a)_{kj} A^j] = (0, 0, +1) ,$$

the previous relation will take the form

$$\sigma^3 = - n_j O_{jl}(\hat{w}) O_{lk}^{-1}(a) \sigma^k . \quad (\text{A.39})$$

Now, let us multiply eq. (A.39) by $B(\hat{w})$ from the left and by $B^{-1}(\hat{w})$ from the right:

$$B(\hat{w})\sigma^3 B^{-1}(\hat{w}) = -n_j O_{jl}(\hat{w}) O_{lk}^{-1}(a) B(\hat{w})\sigma^k B^{-1}(\hat{w}) . \quad (\text{A.40})$$

from where it follows

$$B(\hat{w})\sigma^3 B^{-1}(\hat{w}) = - n_j O_{jl}(\hat{w}) O_{lk}^{-1}(a) O_{ki}^{-1}(\hat{w})\sigma^i , \quad (\text{A.41})$$

or

$$B(\hat{w})\sigma^3 B^{-1}(\hat{w}) = -\sigma^j [O_{ik}(\hat{w}) O_{kl}(a) O_{lj}^{-1}(\hat{w})] n_j . \quad (\text{A.42})$$

In index-free form it looks as

$$B(\hat{w})\sigma^3 B^{-1}(\hat{w}) = -\vec{\sigma} [O(\hat{w}) O(a) O^{-1}(\hat{w})] \vec{n} . \quad (\text{A.43})$$

Relation (A.43), with the use of designation

$$\vec{n}' \equiv [O(\hat{w}) O(a) O^{-1}(\hat{w})] \vec{n} , \quad (\text{A.44})$$

will take the form required:

$$B(\hat{w})\sigma^3 B^{-1}(\hat{w}) = - \vec{\sigma} \vec{n}' . \quad (\text{A.45})$$

It is evident that both definitions of (w) – on the base of (A.45) or (A.27):

$$B(\hat{w}) [\vec{A} \vec{\sigma}] B^{-1}(\hat{w}) = - \vec{n} \vec{\sigma} . \quad (\text{A.46})$$

are absolutely. equivalent. However, the form (A.45) distinctly reveals geometrical meaning of existence a great many of Kustaanheimo-Stifel's coordinates in place of a single one.

Appendix B.

On unitary gauges of spinors

Normalized pseudo vector \vec{n} , $\vec{n}^2 = +1$ can be formed up from a normalized spinor Ψ in accordance with

$$\begin{aligned} \Psi \otimes \Psi^* &= \frac{1}{2}(1 + \sigma^n \vec{n}) , \\ 1 &= \text{sp} (\Psi \otimes \Psi^*) = \Psi^+ \Psi = \text{inv} , \\ n_j &= \text{sp} [\sigma^j (\Psi \otimes \Psi^*)] = \Psi^+ \sigma^j \Psi . \end{aligned}$$

The spinor Ψ corresponding to \vec{n} is given by

$$\Psi = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{1+n_3} e^{-i\gamma/2} \\ \sqrt{1-n_3} e^{+i\gamma/2} \end{vmatrix},$$

$$e^{i\gamma} = \frac{n_1 + in_2}{\sqrt{n_1^2 + n_2^2}}, \quad \gamma \in [-2\pi, +2\pi]. \quad (\text{B.1})$$

At two points of the sphere, $(0,0,+1)$ and $(0,0,-1)$, the spinor has peculiarity – it is not single-valued function of points on the sphere:

$$\Psi^{(+)} = \begin{vmatrix} e^{-i\Gamma/2} \\ 0 \end{vmatrix}, \quad \Psi^{(-)} = \begin{vmatrix} 0 \\ e^{+i\Gamma/2} \end{vmatrix}, \quad (\text{B.2})$$

By definition, $(+)$ -unitary gauge of a vector \vec{n} is result of rotating the vector to the positive axis z :

$$\vec{n} = (n_1, n_2, n_3) \xrightarrow{O(c)} \vec{n}_{(+)} = (0, 0, +1). \quad (\text{B.3})$$

By definition, $(-)$ -unitary gauge of a vector \vec{n} is result of rotating the vector to the negative axis z :

$$\vec{n} = (n_1, n_2, n_3) \xrightarrow{O(c)} \vec{n}_{(-)} = (0, 0, -1). \quad (\text{B.4})$$

There may be defined a counterpart of this concept in spinor formalism:

$(+)$ -unitary gauge of a spinor Ψ is result of $SU(2)$ -rotating the spinor Ψ to the form

$$\Psi = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{1+n_3} e^{-i\gamma/2} \\ \sqrt{1-n_3} e^{+i\gamma/2} \end{vmatrix} \xrightarrow{B(c)} \Psi^{(+)} = \begin{vmatrix} e^{-i\Gamma/2} \\ 0 \end{vmatrix}; \quad (\text{B.5})$$

$(-)$ -unitary gauge of a spinor Ψ is result of $SU(2)$ -rotating the spinor Ψ to the form

$$\Psi = \frac{1}{\sqrt{2}} \begin{vmatrix} \sqrt{1+n_3} e^{-i\gamma/2} \\ \sqrt{1-n_3} e^{+i\gamma/2} \end{vmatrix} \xrightarrow{B(c)} \Psi^{(-)} = \begin{vmatrix} 0 \\ e^{+i\Gamma/2} \end{vmatrix}. \quad (\text{B.6})$$

First, let us consider the case of $(+)$ -unitary gauge of spinor; from which its vector form can be readily produced. The main equation to solve is

$$B(c)\Psi = \Psi^{(+)} . \quad (\text{B.7})$$

In real-valued Kustaanheimo-Stiefel representation this equation takes on the form

$$\begin{vmatrix} c_4 & -c_1 & c_2 & c_3 \\ c_1 & c_4 & c_3 & -c_2 \\ -c_2 & -c_3 & c_4 & -c_1 \\ -c_3 & c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} 0 \\ +\cos(\Gamma/2) \\ -\sin(\Gamma/2) \\ 0 \end{vmatrix}.$$

(B.8)

The problem is to find c_a at given u_a and $e^{i\Gamma/2}$. Equation (B.8) can be re-written as

$$\begin{vmatrix} c_4 & -c_1 & c_2 & c_3 \\ c_1 & c_4 & c_3 & -c_2 \\ -c_2 & -c_3 & c_4 & -c_1 \\ -c_3 & c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} \cos \frac{\Gamma}{2} & 0 & 0 & \sin \frac{\Gamma}{2} \\ 0 & \cos \frac{\Gamma}{2} & \sin \frac{\Gamma}{2} & 0 \\ 0 & -\sin \frac{\Gamma}{2} & \cos \frac{\Gamma}{2} & 0 \\ -\sin \frac{\Gamma}{2} & 0 & 0 & \cos \frac{\Gamma}{2} \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}; \quad (\text{B.9})$$

or in matrix form

$$\begin{aligned} S(c) u &= S \left(\cos \frac{\Gamma}{2}, 0, 0, \sin \frac{\Gamma}{2} \right) u_0^{(+)} , \\ B(c) \Psi &= B \left(\cos \frac{\Gamma}{2}, 0, 0, \sin \frac{\Gamma}{2} \right) \Psi_0^{(+)} . \end{aligned} \quad (\text{B.10})$$

From here it follows

$$\begin{aligned} S \left(\cos \frac{\Gamma}{2}, 0, 0, -\sin \frac{\Gamma}{2} \right) S(c) u &= u_0^{(+)} , \\ B \left(\cos \frac{\Gamma}{2}, 0, 0, -\sin \frac{\Gamma}{2} \right) B(c) \Psi &= \psi_0^{(+)} \end{aligned} \quad (\text{B.11})$$

or

$$S(a) u = u_0^{(+)} , \quad B(a) \Psi = \Psi_0^{(+)} . \quad (\text{B.12})$$

The designation is used:

$$\begin{aligned} S(a) &= S \left(\cos \frac{\Gamma}{2}, 0, 0, -\sin \frac{\Gamma}{2} \right) S(c) , \\ B(a) &= B \left(\cos \frac{\Gamma}{2}, 0, 0, -\sin \frac{\Gamma}{2} \right) B(c) ; \end{aligned} \quad (\text{B.13})$$

or in explicit form

$$\begin{aligned} a_1 &= \cos \frac{\Gamma}{2} c_1 + \sin \frac{\Gamma}{2} c_2 , & a_2 &= -\sin \frac{\Gamma}{2} c_1 + \cos \frac{\Gamma}{2} c_2 , \\ a_3 &= \cos \frac{\Gamma}{2} c_3 - \sin \frac{\Gamma}{2} c_4 , & a_4 &= \sin \frac{\Gamma}{2} c_3 + \cos \frac{\Gamma}{2} c_4 . \end{aligned} \quad (\text{B.14})$$

Equation (B.12) gives

$$\begin{vmatrix} a_4 & -a_1 & a_2 & a_3 \\ a_1 & a_4 & a_3 & -a_2 \\ -a_2 & -a_3 & a_4 & -a_1 \\ -a_3 & a_2 & a_1 & a_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} . \quad (\text{B.15})$$

After ordering it with respect to unknown (a_4, \vec{a}) , it looks as a linear system

$$\begin{vmatrix} u_4 & -u_1 & u_2 & u_3 \\ u_1 & u_4 & -u_3 & u_2 \\ u_2 & -u_3 & -u_4 & -u_1 \\ u_3 & u_2 & u_1 & -u_4 \end{vmatrix} \begin{vmatrix} a_4 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}. \quad (\text{B.16})$$

Its main determinant equals +1:

$$\Delta = (u_1^2 + u_2^2 + u_3^2 + u_4^2)^2 = +1 ;$$

therefore a single solution exists:

$$\begin{aligned} a_4 &= +u_1, & a_1 &= +u_4, \\ a_2 &= -u_3, & a_3 &= +u_2, \end{aligned} \quad (\text{B.17})$$

which is readily checked by (see (B.15))

$$\begin{vmatrix} u_1 & -u_4 & -u_3 & u_2 \\ u_4 & u_1 & u_2 & u_3 \\ u_3 & -u_2 & u_1 & -u_4 \\ -u_2 & -u_3 & u_4 & u_1 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}. \quad (\text{B.18})$$

The solution obtained after translating to spinor form becomes

$$\begin{aligned} B(a) &= u_1 - i\sigma^1 u_4 + i\sigma^1 u_3 - i\sigma^3 u_2, & B(a)\Psi &= \Psi_0^+, \\ \begin{vmatrix} u_1 - iu_2 & u_3 - iu_4 \\ -u_3 - iu_4 & u_1 + iu_2 \end{vmatrix} \begin{vmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{vmatrix} &= \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \end{aligned}$$

Thus, the problem of (+)-unitary gauge of spinors is solved (with eq. (B.11)):

$$\begin{aligned} B(a)\Psi &= \Psi_0^{(+)}, & B(a) &= B\left(\cos\frac{\Gamma}{2}, 0, 0, -\sin\frac{\Gamma}{2}\right) B(c), \\ B(c)\Psi &= \Psi^+, & B(c) &= B\left(\cos\frac{\Gamma}{2}, 0, 0, \sin\frac{\Gamma}{2}\right) B(a). \end{aligned} \quad (\text{B.19})$$

Explicit form of (c) is

$$\begin{aligned} c_1 &= \cos\frac{\Gamma}{2} u_4 + \sin\frac{\Gamma}{2} u_3, & c_2 &= \sin\frac{\Gamma}{2} u_4 - \cos\frac{\Gamma}{2} u_3, \\ c_3 &= \cos\frac{\Gamma}{2} u_2 + \sin\frac{\Gamma}{2} u_1, & c_4 &= -\sin\frac{\Gamma}{2} u_2 + \cos\frac{\Gamma}{2} u_1 \end{aligned} \quad (\text{B.20})$$

and $B(c)$ looks

$$B(c) = \begin{vmatrix} e^{-i\Gamma/2}(u_1 - iu_2) & e^{-i\Gamma/2}(u_3 - iu_4) \\ -e^{+i\Gamma/2}(u_1 + iu_2) & e^{+i\Gamma/2}(u_1 + iu_2) \end{vmatrix}$$

which can be readily verified by

$$\left| \begin{array}{cc} e^{-i\Gamma/2} (u_1 - i u_2) & e^{-i\Gamma/2} (u_3 - i u_4) \\ -e^{+i\Gamma/2} (u_1 + i u_2) & e^{+i\Gamma/2} (u_3 + i u_4) \end{array} \right| \left| \begin{array}{c} u_1 + i u_2 \\ u_3 + i u_4 \end{array} \right| = \left| \begin{array}{c} e^{-i\Gamma/2} \\ 0 \end{array} \right|.$$

It is simple thing to show how this theory of unitary spinor gauge is referred to the well-known theory of flat orthogonal rotations translating the vector $\vec{n} = (n_1, n_2, n_3)$ into $\vec{n}_{(+)} = (0, 0, +1)$ (see [59,60]). At the first place it should be noted that presence of a phase factor $e^{-i\Gamma/2}$ at Ψ^+ by no means influences the vector associated with this spinor:

$$\begin{aligned} \Psi^+ &\implies \vec{n}_{(+)} = (0, 0, +1), \\ \Psi_0^+ &\implies \vec{n}_{(+)} = (0, 0, +1). \end{aligned} \quad (\text{B.21})$$

This circumstance may be used as follows: let this phase be such that c_3 vanishes:

$$c_3 = 0 \implies \cos \frac{\Gamma}{2} u_2 + \sin \frac{\Gamma}{2} u_1 = 0, \quad (\text{B.22})$$

that is

$$\tan \frac{\Gamma}{2} = - \frac{u_2}{u_1}, \quad \frac{\Gamma}{2} \in [-\pi, +\pi].$$

Such a special choice of the phase does not contain any puzzle. Indeed, out of the axis z for real-valued u_1 and u_2 we have expressions

$$u_1 = \sqrt{1 - n_3} \cos \frac{\gamma}{2}, \quad u_2 = -\sqrt{1 - n_3} \sin \frac{\gamma}{2}. \quad (\text{B.23})$$

Now preserving γ constant one can change $n_3 \rightarrow +1$, so that the variable γ turns out to be a "mute" coordinate Γ :

$$\Gamma = \gamma. \quad (\text{B.24})$$

In other word, the choice (B.22) is simple acceptance of equations (B.23)–(B.24).

With (B.22), expression for a spinor matrix $B(c)$ becomes simpler, and what is more, its vector counterpart, orthogonal matrix $O(c)$ is much simplified as well. Indeed,

$$\begin{aligned} c_3 &= 0, \quad c_4 = \cos(\Gamma/2) \frac{u_1^2 + u_2^2}{u_1}, \\ c_1 &= \cos(\Gamma/2) \frac{u_4 u_1 - u_2 u_3}{u_1}, \quad c_2 = -\cos(\Gamma/2) \frac{u_3 u_1 + u_2 u_3}{u_1}, \end{aligned} \quad (\text{B.25})$$

3-vector parameter becomes

$$\begin{aligned} C_1 &= \frac{c_1}{c_4} = + \frac{u_4 u_1 - u_2 u_3}{u_1^2 + u_2^2} = + \frac{n_2}{1 + n_3}, \\ C_2 &= \frac{c_2}{c_4} = - \frac{u_1 u_3 + u_2 u_4}{u_1^2 + u_2^2} = - \frac{n_1}{1 + n_3}, \quad C_3 = 0. \end{aligned} \quad (\text{B.26})$$

The obtained $C_i(n)$ is the well-known expression for 3-vector parameter of a flat rotation from $SO(3.R)$, translating vector \vec{n} onto positive axis z . This \vec{C} is directed along

$$\vec{n} \times \vec{n}_{(+)} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & +1 \end{vmatrix} = n_2 \vec{e}_1 - n_1 \vec{e}_2.$$

With the help of

$$n_1 = \sin \theta \cos \phi, \quad n_2 = \sin \theta \sin \phi, \quad n_3 = \cos \theta,$$

the modulus of the vector is found immediately:

$$|\vec{C}| = \sqrt{\frac{n_1^2 + n_2^2}{(1 + n_3)^2}} = \sqrt{\frac{\sin^2 \theta}{(1 + \cos \theta)^2}} = \tan \frac{\theta}{2}.$$

In the same line, one can consider $(-)$ -unitary spinor and vector gauges; at this the most calculation need not do be repeated. The principal formulas are:

$$\begin{aligned} \Psi &= \begin{vmatrix} u_1 + i u_2 \\ u_3 + i u_4 \end{vmatrix}, \quad \Psi^{(-)} = \begin{vmatrix} 0 \\ e^{+i\Gamma/2} \end{vmatrix}, \quad \Psi_0^{(-)} = \begin{vmatrix} 0 \\ 1 \end{vmatrix}, \\ B(c)\Psi &= \Psi^{(-)}, \quad \begin{vmatrix} c_4 & -c_1 & c_2 & c_3 \\ c_1 & c_4 & c_3 & -c_2 \\ -c_2 & -c_3 & c_4 & -c_1 \\ -c_3 & c_2 & c_1 & c_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} \sin(\Gamma/2) \\ 0 \\ 0 \\ \cos(\Gamma/2) \end{vmatrix}, \\ &B\left(\cos \frac{\Gamma}{2}, 0, 0, + \sin \frac{\Gamma}{2}\right) \Psi_0^{(-)} = \Psi^{(-)}, \\ &B\left(\cos \frac{\Gamma}{2}, 0, 0, - \sin \frac{\Gamma}{2}\right) B(c) \Psi = \Psi_0^{(-)}, \quad B(a) \Psi = \Psi_0^{(-)}. \end{aligned} \quad (\text{B.27})$$

Equations from (B.27) after translating to real-valued form will look

$$\begin{vmatrix} a_4 & -a_1 & a_2 & a_3 \\ a_1 & a_4 & a_3 & -a_2 \\ -a_2 & -a_3 & a_4 & -a_1 \\ -a_3 & a_2 & a_1 & a_4 \end{vmatrix} \begin{vmatrix} u_4 \\ u_1 \\ u_2 \\ u_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix};$$

which after re-grouping it with respect to variables (a_4, \vec{a}) looks

$$\begin{vmatrix} u_4 & -u_1 & u_2 & u_3 \\ u_1 & u_4 & -u_3 & u_2 \\ u_2 & -u_3 & -u_4 & -u_1 \\ u_3 & u_2 & u_1 & -u_4 \end{vmatrix} \begin{vmatrix} a_4 \\ a_1 \\ a_2 \\ a_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}; \quad (\text{B.28})$$

its solution

$$a_4 = +u_3, \quad a_1 = +u_2, \quad a_2 = +u_1, \quad a_3 = -u_4. \quad (\text{B.29})$$

Corresponding (c_4, \vec{c}) :

$$\begin{aligned} c_1 &= \cos \frac{\Gamma}{2} u_2 - \sin \frac{\Gamma}{2} u_1 , & c_2 &= \sin \frac{\Gamma}{2} u_2 + \cos \frac{\Gamma}{2} u_1 , \\ c_3 &= -\cos \frac{\Gamma}{2} u_4 + \sin \frac{\Gamma}{2} u_3 , & c_4 &= \sin \frac{\Gamma}{2} u_4 + \cos \frac{\Gamma}{2} u_3 \end{aligned} \quad (\text{B.30})$$

spinor matrix $B(c)$:

$$B = \begin{vmatrix} e^{-i\Gamma/2}(u_3 + iu_4) & -e^{-i\Gamma/2}(u_1 + iu_2) \\ e^{+i\Gamma/2}(u_1 - iu_2) & e^{+i\Gamma/2}(u_3 - iu_4) \end{vmatrix} ;$$

gives equation $B(c)\Psi = \Psi^{(-)}$ which is checked by

$$\begin{vmatrix} e^{-i\Gamma/2}(u_3 + iu_4) & -e^{-i\Gamma/2}(u_1 + iu_2) \\ e^{+i\Gamma/2}(u_1 - iu_2) & e^{+i\Gamma/2}(u_3 - iu_4) \end{vmatrix} \begin{vmatrix} u_1 + iu_2 \\ u_3 + iu_4 \end{vmatrix} = \begin{vmatrix} 0 \\ e^{+i\Gamma/2} \end{vmatrix} .$$

On special choice of Γ :

$$c_3 = 0 : \implies -\cos \frac{\Gamma}{2} u_4 + \sin \frac{\Gamma}{2} u_3 = 0, \quad (\text{B.31})$$

that is

$$\tan \frac{\Gamma}{2} = \frac{u_4}{u_3} ,$$

expressions for $B(c)$ and $O(c)$ are much simplified:

$$\begin{aligned} c_1 &= \cos \frac{\Gamma}{2} \frac{u_2 u_3 - u_4 u_3}{u_3} , & c_2 &= \cos \frac{\Gamma}{2} \frac{u_1 u_3 + u_2 u_4}{u_3} , \\ c_3 &= 0 , & c_4 &= \cos \frac{\Gamma}{2} \frac{u_3^2 + u_4^2}{u_3} , \end{aligned} \quad (\text{B.32})$$

3-vector parameter is

$$\begin{aligned} C_1 &= \frac{c_1}{c_4} = \frac{u_2 u_2 - u_4 u_1}{u_3^2 + u_4^2} = -\frac{n_2}{1 - n_3} , \\ C_2 &= \frac{c_2}{c_4} = \frac{u_1 u_3 + u_2 u_4}{u_3^2 + u_4^2} = \frac{n_1}{1 - n_3} , & C_3 &= 0 . \end{aligned} \quad (\text{B.33})$$

this \vec{C} is directed along

$$\vec{n} \times \vec{n}_{(-)} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & -1 \end{vmatrix} = -n_2 \vec{e}_1 + n_1 \vec{e}_2 .$$

With

$$n_1 = \sin \theta \cos \phi , \quad n_2 = \sin \theta \sin \phi , \quad n_3 = \cos \theta ,$$

for modulus of the vector we get

$$|\vec{C}| = +\sqrt{\frac{n_1^2 + n_2^2}{(1 - n_3)^2}} = \tan\left(\frac{\pi - \theta}{2}\right).$$

GENERALIZATION

Let us find $SU(2)$ -rotation, transforming spinor Ψ into Ψ' :

$$B(c) \Psi = \Psi', \quad (\text{B.34})$$

The task may be solve by two ways. First, let us consider eq. (B.34) in its real-valued form (compare with (B.28))

$$\begin{vmatrix} u_4 & -u_1 & u_2 & u_3 \\ u_1 & u_4 & -u_3 & u_2 \\ u_2 & -u_3 & -u_4 & -u_1 \\ u_3 & u_2 & u_1 & -u_4 \end{vmatrix} \begin{vmatrix} c_4 \\ c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} u'_4 \\ u'_1 \\ u'_2 \\ u'_3 \end{vmatrix}, \quad (\text{B.35})$$

its solution is

$$\begin{aligned} c_4 &= u'_4 u_4 + u'_1 u_1 + u'_2 u_2 + u'_3 u_3, \\ c_1 &= u'_1 u_4 - u'_4 u_1 - u'_2 u_3 + u'_3 u_2, \\ c_2 &= -u'_2 u_4 + u'_4 u_2 + u'_3 u_1 - u'_3 u_1, \\ c_3 &= -u'_3 u_4 + u'_4 u_3 + u'_1 u_2 - u'_2 u_1. \end{aligned} \quad (\text{B.36})$$

Else one, independent, more simple and symmetric consideration is based on describing of normalized spinor by unitary 2×2 matrices, when action of $SU(2)$ -rotation is given by

$$\begin{aligned} B(\hat{c}) B(u) &= B(u') \implies \\ B(\hat{c}) &= B(u') B^{-1}(u) = B(u') B(\bar{u}). \end{aligned} \quad (\text{B.37})$$

Result (B.37) coincides with (B.36); indeed, eq.(B.36) is written as

$$\begin{aligned} \hat{c}_4 &= u'_4 \bar{u}_4 - u'_1 \bar{u}_1 - u'_2 \bar{u}_2 - u'_3 \bar{u}_3, \\ \hat{c}_1 &= u'_1 \bar{u}_4 + u'_4 \bar{u}_1 + u'_2 \bar{u}_3 - u'_3 \bar{u}_2, \\ \hat{c}_2 &= u'_2 \bar{u}_4 + u'_4 \bar{u}_2 + u'_3 \bar{u}_1 - u'_3 \bar{u}_1, \\ \hat{c}_3 &= u'_3 \bar{u}_4 + u'_4 \bar{u}_3 + u'_1 \bar{u}_2 - u'_2 \bar{u}_1, \end{aligned} \quad (\text{B.38})$$

or

$$\hat{c}_4 = u'_4 \bar{u}_4 - u'_i \bar{u}_i, \quad \hat{c}_i = u'_i \bar{u}_4 + u'_4 \bar{u}_i + \epsilon_{ijk} u'_j \bar{u}_k. \quad (\text{B.39})$$

One principal point should be noted: each of spinor equations

$$B(c) \Psi = \pm \Psi, \quad (\text{B.40})$$

has only trivial solution; indeed, eqs. (B.36) give

$$\begin{aligned} c_4 &= \pm(u_4 u_4 + u_1 u_1 + u_2 u_2 + u_3 u_3) = \pm 1 , \\ c_1 &= \pm(u_1 u_4 - u_4 u_1 - u_2 u_3 + u_3 u_2) = 0 , \\ c_2 &= \pm(-u_2 u_4 + u_4 u_2 + u_3 u_1 - u_3 u_1) = 0 , \\ c_3 &= \pm(-u_3 u_4 + u_4 u_3 + u_1 u_2 - u_2 u_1) = 0 , \end{aligned}$$

that is

$$B(c) = \pm I . \quad (\text{B.41})$$

In other words, turning to spinor description eliminates from the formalism the concept of small group: in the case of vectors it is isomorphic to $SO(2)$, in the case of spinors it is reduced to trivial group of a single element, identity I .

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